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All supersymmetric solutions of $N = 2$, $D = 4$ gauged supergravity

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Abstract: We classify all supersymmetric solutions of minimal gauged supergravity in four dimensions. There are two classes of solutions that are distinguished by the norm of the Killing vector constructed from the Killing spinor. If the Killing vector is timelike, the solutions are determined by the geometry of a two-dimensional base-manifold. When it is lightlike, the most general BPS solution is given by an electrovac AdS travelling wave. This supersymmetric configuration was previously unknown. Generically the solutions preserve one quarter of the supersymmetry. Also in the timelike case we show that there exist new BPS solutions, which are of Petrov type I, and are thus more general than the previously known type D configurations. These geometries can be uplifted to obtain new solutions of eleven-dimensional supergravity.

Keywords: Superstring Vacua, Black Holes, Supergravity Models.

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1. Introduction

Supersymmetric solutions to supergravity theories have played, and continue to play, an important role in developments in string theory. This makes it desirable to obtain a complete classification of BPS solutions to various supergravities in diverse dimensions. Progress in this direction has been made recently using the mathematical notion of G-structures [1]. The basic strategy is to assume the existence of at least one Killing spinor, and to construct differential forms as bilinears from this Killing spinor. These forms, which define a preferred G-structure, obey several algebraic and differential equations that can be used to deduce the metric and the other bosonic supergravity fields. Despite some partial progress [2, 3], a complete classification of supersymmetric geometries in eleven- and ten-dimensional supergravities seems to be currently out of reach. This motivates to consider simpler, lower-dimensional supergravities, in particular the ones that can be obtained by Kaluza-Klein reduction from ten or eleven dimensions. A systematic classification of supersymmetric solutions has been obtained for minimal supergravity in five dimensions, both in the ungauged [4] and gauged case [5], and for minimal supergravity in six dimensions [6].

Here we present a complete classification of BPS solutions in minimal gauged four-dimensional supergravity. Using the techniques of [7] to uplift the geometries to eleven dimensions, our analysis can be viewed as a classification of a restricted class of eleven-dimensional solutions. The most general configuration admitting Killing spinors in the ungauged four-dimensional case was obtained by Tod many years ago [8], generalizing earlier work by Gibbons and Hull [9]. In the absence of dust sources, the resulting geometry is given by an Israel-Wilson-Perjes metric if the Killing vector constructed from the Killing spinor is timelike, and by a plane-wave spacetime if this Killing vector is lightlike. In the gauged case, we will find more complicated BPS solutions: In the timelike case, they are determined by the geometry of a two-dimensional base manifold, whereas in the lightlike case they are given by electrovac AdS travelling waves. A further new feature is the fact that generically the solutions preserve only one quarter of the supersymmetry, whereas in the ungauged case they are one half supersymmetric [8].

Supersymmetric solutions to minimal gauged supergravity in four dimensions have been studied in [10]–[15]. Among these are the one quarter supersymmetric magnetic monopoles [10, 12] that have no well-defined limit when the gauge coupling constant goes to zero, because their magnetic charge is quantized in terms of the inverse coupling constant. Apart from the pp-waves in AdS considered in [15], which have Petrov type N, the BPS solutions that were hitherto known do all belong to a subclass of the most general Petrov type D metric found by Plebański and Demiański [16], which is determined by electric and magnetic charges, nut charge, and mass and rotation parameters. The conditions under which this metric admits Killing spinors were obtained in [14]. We shall use our formalism to show that there exist also BPS solutions that have Petrov type I, and therefore do not belong to the classes mentioned above.

The remainder of this paper is organized as follows: in section 2, we briefly review minimal gauged supergravity in four dimensions, and obtain the algebraic and differential constraints satisfied by the differential forms constructed as bilinears in the Killing spinor. In section 3 these constraints are solved for the case in which the Killing vector resulting from the Killing spinor is timelike, and the metric and electromagnetic field strength are deduced. As specific examples of the general BPS solution, we recover the one quarter supersymmetric Reissner-Nordström-Taub-NUT-AdS spacetime obtained in [14], and construct a previously unknown supersymmetric Petrov type I configuration. In section 4, the lightlike case is analyzed, and it is shown that the most general supersymmetric geometry is given by an electrovac AdS travelling wave. In section 5 we show that the only maximally supersymmetric solution is AdS_4 with vanishing gauge field. We conclude with some final remarks in section 6. The appendix contains our conventions and some useful identities.

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2. $N = 2$, $D = 4$ gauged supergravity

The gauged version of $N = 2$ supergravity was found by Freedman and Das [17] and by Fradkin and Vasiliev [18]. In this theory, the rigid SO(2) symmetry rotating the two independent Majorana supersymmetries present in the ungauged theory, is made local by introduction of a minimal gauge coupling between the photons and the gravitini. Local supersymmetry then requires a negative cosmological constant and a gravitini mass term. The theory has four bosonic and four fermionic degrees of freedom; it describes a graviton e^a_μ, two real gravitini ψ^i_μ $(i = 1, 2)$, and a Maxwell gauge field $\mathcal{A}_\mu$. As we said, the latter is minimally coupled to the gravitini, with coupling constant ℓ^{-1}. If we combine the two ψ^i_μ to a single complex spinor $\psi_\mu = \psi^1_\mu + i\psi^2_\mu$, the lagrangian reads (cf. also [10])

$$\begin{aligned} e^{-1}L = &-\frac{1}{4}R + \frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} - \frac{3}{2\ell^2} + \frac{1}{2}\bar{\psi}_\mu\Gamma^{\mu\nu\rho}D_\nu\psi_\rho + \\ &+ \frac{i}{8}(\mathcal{F}^{\mu\nu} + \hat{\mathcal{F}}^{\mu\nu})\bar{\psi}_\rho\Gamma_{[\mu}\Gamma^{\rho\sigma}\Gamma_{\nu]}\psi_\sigma - \frac{1}{2\ell}\bar{\psi}_\mu\Gamma^{\mu\nu}\psi_\nu\,. \end{aligned} \tag{2.1}$$

We see that the cosmological constant is $\Lambda = -3\ell^{-2}$. D_μ denotes the gauge- and Lorentz-covariant derivative defined by

$$D_\mu = \nabla_\mu - i\ell^{-1}\mathcal{A}_\mu\,, \tag{2.2}$$

where ∇_μ is the Lorentz-covariant derivative

$$\nabla_\mu = \partial_\mu + \frac{1}{4}\omega_\mu{}^{ab}\Gamma_{ab}\,. \tag{2.3}$$

The equation of motion for the spin connection $\omega_\mu{}^{ab}$ reads

$$\omega_{\mu ab} = \Omega_{\mu ab} - \Omega_{\mu ba} - \Omega_{ab\mu}\,, \tag{2.4}$$

where

$$\Omega_{\mu\nu}{}^a \equiv \partial_{[\mu}\, e_{\nu]}^{\;a} - \frac{1}{2}\mathrm{Re}(\bar{\psi}_\mu\Gamma^a\psi_\nu)\,. \tag{2.5}$$

$\hat{\mathcal{F}}_{\mu\nu}$ denotes the supercovariant field strength given by

$$\hat{\mathcal{F}}_{\mu\nu} = \mathcal{F}_{\mu\nu} - \mathrm{Im}(\bar{\psi}_\mu\psi_\nu)\,. \tag{2.6}$$

The action is invariant under the local supersymmetry transformations

$$\begin{aligned} \delta e^a_\mu &= \mathrm{Re}(\bar{\epsilon}\Gamma^a\psi_\mu)\,, \\ \delta\mathcal{A}_\mu &= \mathrm{Im}(\bar{\epsilon}\psi_\mu)\,, \\ \delta\psi_\mu &= \mathcal{D}_\mu\epsilon\,. \end{aligned} \tag{2.7}$$

In (2.7) ϵ is an infinitesimal Dirac spinor, and $\mathcal{D}_\mu$ is the supercovariant derivative defined by

$$\mathcal{D}_\mu = D_\mu + \frac{1}{2\ell}\Gamma_\mu + \frac{i}{4}\hat{\mathcal{F}}_{ab}\Gamma^{ab}\Gamma_\mu\,. \tag{2.8}$$

Invariance of bosonic backgrounds under the local supersymmetry transformations (2.7) yields the Killing spinor equation

$$\mathcal{D}_\mu\epsilon = 0\,. \tag{2.9}$$

2.1 Differential forms constructed from the killing spinor

Any supersymmetric geometry must admit at least one Killing spinor ϵ, from which we can construct the following bosonic differential forms:

- a scalar $f = \bar{\epsilon}\epsilon$,
- a pseudo-scalar $g = i\bar{\epsilon}\Gamma_5\epsilon$,
- a vector $V_\mu = i\bar{\epsilon}\Gamma_\mu\epsilon$,
- a pseudo-vector $A_\mu = i\bar{\epsilon}\Gamma_5\Gamma_\mu\epsilon$,
- an antisymmetric tensor $\Phi_{\mu\nu} = i\bar{\epsilon}\Gamma_{\mu\nu}\epsilon$.

The i factors have been inserted in such a way that all these quantities are real. As the matrices $\mathbf{1}$, Γ_a, $\Gamma_5\Gamma_a$ and Γ_{ab} form a basis of the space of 4×4 matrices, any other differential form constructed from the spinor ϵ can be written as a linear combination of the previous ones.

In all, we have 16 real components, which cannot be all independent since an arbitrary spinor has real dimension 8. There must thus exist some algebraic relations between the differential forms, which can be obtained using the Fierz identity. From the spinor ϵ, we can construct a matrix $\epsilon\bar{\epsilon}$, which has as components in the Γ-basis[1]

$$4\epsilon\bar{\epsilon} = f\cdot\mathbf{1} - iV^\mu\Gamma_\mu + \frac{i}{2}\Phi^{\mu\nu}\Gamma_{\mu\nu} + iA^\mu\Gamma_5\Gamma_\mu - ig\Gamma_5\,. \tag{2.10}$$

As a consequence,

$$iV^\mu\Gamma_\mu\epsilon = -iA^\mu\Gamma_5\Gamma_\mu\epsilon = -\left(f + ig\Gamma_5\right)\epsilon \tag{2.11}$$

and

$$i\Phi^{\alpha\beta}\Gamma_{\alpha\beta}\epsilon = 2\left(f - ig\Gamma_5\right)\epsilon\,. \tag{2.12}$$

Contracting these relations with $\bar{\epsilon}$ we obtain

$$N = V^2 = -A^2 = -\left(f^2+g^2\right), \qquad f^2 - g^2 = \frac{1}{2}\Phi^2\,, \tag{2.13}$$

where we have defined N as the norm squared of V. Hence $N \le 0$ and it follows that V is either timelike (and A spacelike) or lightlike. A contraction with $\bar{\epsilon}\Gamma_5$ yields $V\cdot A = 0$. The other relations which can be found from the Fierz identity are

$$fV_\mu = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}A^\nu\Phi^{\rho\sigma}\,, \qquad fA_\mu = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}V^\nu\Phi^{\rho\sigma}\,, \tag{2.14}$$

$$gV_\mu = \Phi_{\mu\nu}A^\nu\,, \qquad gA_\mu = \Phi_{\mu\nu}V^\nu\,, \tag{2.15}$$

$$fg = -\frac{1}{8}\varepsilon_{\mu\nu\rho\sigma}\Phi^{\mu\nu}\Phi^{\rho\sigma}\,, \tag{2.16}$$

$$f\Phi_{\mu\nu} = -\varepsilon_{\mu\nu\rho\sigma}V^\rho A^\sigma + \frac{1}{2}g\varepsilon_{\mu\nu\rho\sigma}\Phi^{\rho\sigma}\,, \tag{2.17}$$

$$\Phi_{(\mu}{}^{\rho}\varepsilon_{\nu)\rho\alpha\beta}\Phi^{\alpha\beta} - \frac{1}{4}g_{\mu\nu}\varepsilon_{\rho\sigma\alpha\beta}\Phi^{\rho\sigma}\Phi^{\alpha\beta} = 0\,. \tag{2.18}$$

[1] Any matrix M can be expanded as $4M = \mathbf{1}\cdot\mathrm{tr}\,M + \Gamma_\mu\,\mathrm{tr}(M\Gamma^\mu) - \frac{1}{2}\Gamma_{\mu\nu}\,\mathrm{tr}(M\Gamma^{\mu\nu}) - \Gamma_5\Gamma_\mu\,\mathrm{tr}(M\Gamma_5\Gamma^\mu) + \Gamma_5\,\mathrm{tr}(M\Gamma_5)$. Taking $M = \epsilon\bar{\epsilon}$ and using $\mathrm{tr}\,\epsilon K\bar{\epsilon} = \bar{\epsilon}K\epsilon$ for any matrix K we obtain the Fierz identity.

After having obtained the algebraic relations obeyed by the differential forms, let us now derive the differential constraints. Starting from the supercovariant derivative (2.8), one can define a left-acting supercovariant derivative $\overleftarrow{\mathcal{D}}_\mu$ by the relation $\bar{\psi}\overleftarrow{\mathcal{D}}_\mu \equiv \overline{\mathcal{D}_\mu\psi}$, yielding

$$\overleftarrow{\mathcal{D}}_\mu = \overleftarrow{\nabla}_\mu + \frac{i}{\ell}\mathcal{A}_\mu - \frac{1}{2\ell}\Gamma_\mu - \frac{i}{4}\mathcal{F}_{ab}\Gamma_\mu\Gamma^{ab}\,, \tag{2.19}$$

where we have defined the left-acting covariant derivative

$$\overleftarrow{\nabla} \equiv \overleftarrow{\partial} - \frac{1}{4}\omega_\mu{}^{ab}\Gamma_{ab}\,, \tag{2.20}$$

such that $\bar{\psi}\overleftarrow{\nabla}_\mu \equiv \overline{\nabla_\mu\psi}$. With these definitions, we obtain a Leibniz rule for the derivative of the contraction of two spinors

$$\partial_\mu\left(\bar{\psi}\chi\right) = \bar{\psi}\overleftarrow{\nabla}_\mu\chi + \bar{\psi}\overrightarrow{\nabla}_\mu\chi\,. \tag{2.21}$$

With some algebra the previous identity can be generalized for insertions of Γ-matrices:

$$\nabla_\mu\left(\bar{\psi}\Gamma_{\mu_1}\cdots\Gamma_{\mu_n}\chi\right) = \bar{\psi}\overleftarrow{\nabla}_\mu\Gamma_{\mu_1}\cdots\Gamma_{\mu_n}\chi + \bar{\psi}\Gamma_{\mu_1}\cdots\Gamma_{\mu_n}\overrightarrow{\nabla}_\mu\chi\,, \tag{2.22}$$

and with a Γ_5 matrix:

$$\nabla_\mu\left(\bar{\psi}\Gamma_5\Gamma_{\mu_1}\cdots\Gamma_{\mu_n}\chi\right) = \bar{\psi}\overleftarrow{\nabla}_\mu\Gamma_5\Gamma_{\mu_1}\cdots\Gamma_{\mu_n}\chi + \bar{\psi}\Gamma_5\Gamma_{\mu_1}\cdots\Gamma_{\mu_n}\overrightarrow{\nabla}_\mu\chi\,. \tag{2.23}$$

Using these relations as well as $\mathcal{D}_\mu\epsilon = 0$, it is straightforward to show that f, g, V_μ, A_μ and $\Phi_{\mu\nu}$ satisfy the differential constraints

$$\begin{aligned}
\partial_\mu f &= \mathcal{F}_{\mu\nu}V^\nu\,, && (2.24)\\
\partial_\mu g &= -\frac{1}{\ell}A_\mu - \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}V^\nu\mathcal{F}^{\rho\sigma}\,, && (2.25)\\
\nabla_\mu V_\nu &= \frac{1}{\ell}\Phi_{\mu\nu} - f\mathcal{F}_{\mu\nu} + \frac{1}{2}g\varepsilon_{\mu\nu\rho\sigma}\mathcal{F}^{\rho\sigma}\,, && (2.26)\\
\nabla_\mu A_\nu &= -\frac{1}{\ell}gg_{\mu\nu} - \mathcal{F}_{(\mu}{}^\rho\varepsilon_{\nu)\rho\alpha\beta}\Phi^{\alpha\beta} + \frac{1}{4}g_{\mu\nu}\varepsilon_{\rho\sigma\alpha\beta}\mathcal{F}^{\rho\sigma}\Phi^{\alpha\beta}\,, && (2.27)\\
\nabla_\mu\Phi_{\alpha\beta} &= \frac{2}{\ell}g_{\mu[\alpha}V_{\beta]} + 2\mathcal{F}_{[\alpha}{}^\sigma\varepsilon_{\beta]\sigma\mu\nu}A^\nu + \mathcal{F}_\mu{}^\sigma\varepsilon_{\sigma\alpha\beta\nu}A^\nu + g_{\mu[\alpha}\varepsilon_{\beta]\nu\rho\sigma}A^\nu\mathcal{F}^{\rho\sigma}\,. && (2.28)
\end{aligned}$$

Furthermore, taking the symmetric part of (2.26) we obtain $\nabla_{(\mu}V_{\nu)} = 0$, hence V^μ is a Killing vector.

3. The timelike case

3.1 Construction of the bosonic fields

In the timelike case we have $N < 0$, hence V is a timelike vector and A a spacelike vector. Furthermore, the antisymmetric part of (2.27) implies $\mathrm{d}A = 0$, so that locally there exists a function z such that $A = \mathrm{d}z$. Introducing the coordinates (t, x, y, z) such that $V = \partial_t$, and x, y are two transverse coordinates, the metric can be written as

$$\mathrm{d}s^2 = N\,(\mathrm{d}t + \omega)^2 - \frac{1}{N}\left(\mathrm{d}z^2 + h_{ij}\,(\mathrm{d}x^i + a^i\mathrm{d}z)(\mathrm{d}x^j + a^j\mathrm{d}z)\right). \tag{3.1}$$

Here the latin indices $i, j, \ldots$ range from 1 to 2, with $x^1 = x$, $x^2 = y$. As V is Killing, the function N, the one-form ω, the transverse metric h_{ij} as well as a^i are functions of (x, y, z) but are independent of t; the metric is stationary. A contraction of V with equations (2.24) and (2.25) shows that the functions f and g are also time-independent. Note that a^i can be eliminated by a diffeomorphism

$$x^i = x^i(x'^j, z)\,, \qquad \text{with} \qquad \frac{\partial x^i}{\partial z} = -a^i\,, \tag{3.2}$$

so that we can set $a^i = 0$ without loss of generality. The resulting metric is invariant under the transformation

$$t \mapsto t + \chi(x, y, z)\,, \qquad \omega \mapsto \omega - \mathrm{d}\chi\,, \tag{3.3}$$

where χ is an arbitrary function of x, y and z. We can use this freedom to eliminate the ω_z component of the shift vector. Hence, without loss of generality, we can take $\omega = \omega_i\,\mathrm{d}x^i$.

We come now to the two-form Φ and the electromagnetic field strength $\mathcal{F}$. Following appendix B, we decompose Φ with respect to the vector V and, using equations (2.14) and (2.15), we obtain

$$\Phi = -\frac{1}{N}\left[gV \wedge A - f^*\left(V \wedge A\right)\right]. \tag{3.4}$$

Similarly, equations (2.24), (2.25) immediately yield the decomposition of $\mathcal{F}$ with respect to the Killing vector V,

$$\mathcal{F} = -\frac{1}{N}\left[V \wedge \mathrm{d}f + {}^*\left(V \wedge \left(\mathrm{d}g + \frac{1}{\ell}A\right)\right)\right]. \tag{3.5}$$

The electromagnetic field is invariant under the 1-parameter group of transformations generated by the timelike Killing vector V

$$\mathcal{L}_V \mathcal{F} = \mathcal{L}_V{}^*\mathcal{F} = 0\,, \tag{3.6}$$

hence $\mathcal{F}$ is a stationary electromagnetic field, and the full solution is invariant under the time translations generated by V.

In order to determine the shift functions ω_i, we first note that $V = N\,(\mathrm{d}t + \omega)$ as a one-form. Combining its exterior derivative $\mathrm{d}V = \frac{\mathrm{d}N}{N} \wedge V + \mathrm{d}\omega$ with the equation (2.26) for V and substituting Φ and $\mathcal{F}$ using equations (3.4) and (3.5), we obtain

$$\mathrm{d}\omega = \frac{2}{N^2}\;{}^*\left[V \wedge \left(f\mathrm{d}g - g\mathrm{d}f + \frac{2}{\ell}fA\right)\right], \tag{3.7}$$

or, in components,

$$\partial_z \omega_i = \frac{2\sqrt{h}}{N^2} h^{jl} \epsilon_{il}\left(f\partial_j g - g\partial_j f\right)\,, \tag{3.8}$$

$$\partial_i \omega_j - \partial_j \omega_i = \frac{2\sqrt{h}}{N^2}\epsilon_{ij}\left(f\partial_z g - g\partial_z f + \frac{2f}{\ell}\right), \tag{3.9}$$

where $h = \det h_{ij}$ and $\epsilon_{12} = 1$. If the functions f and g are known, these equations completely determine ω_i, up to gauge transformations of the form (3.3).

Next we consider equation (2.27). We first use the expressions (3.4) and (3.5) for Φ and $\mathcal{F}$ in (2.27) and project the resulting equation on A^μ. In this way one gets

$$\nabla_A A_\nu = -\frac{1}{2}\nabla_\nu N\,, \tag{3.10}$$

which is already satisfied since $A^2 = -N$ and $\mathrm{d}A = 0$. Similarly, also the projection on V^μ does not yield anything new, if the eqns. (2.15), (2.26), (3.5) as well as $A \cdot V = 0$ hold. The only nontrivial information comes from the projection on the transverse metric

$$h_{\mu\nu} = -Ng_{\mu\nu} + V_\mu V_\nu - A_\mu A_\nu\,, \tag{3.11}$$

from which we obtain

$$h^\mu{}_\alpha h^\nu{}_\beta \nabla_\mu A_\nu = \frac{2gN}{\ell}h_{\alpha\beta} + \frac{1}{2}h_{\alpha\beta}A^\mu\nabla_\mu N\,. \tag{3.12}$$

This can be viewed as a condition on the extrinsic curvature of the two-surface with metric $h_{\alpha\beta}$ as embedded into the three-manifold $\mathrm{d}z^2 + h_{ij}dx^i dx^j$. Evaluating (3.12) yields

$$\partial_z h_{ij} = \frac{4g}{\ell N}h_{ij}\,, \tag{3.13}$$

with the general solution

$$h_{ij} = C_{ij}(x^k)\,\exp\left(\int \frac{4g}{\ell N}\,\mathrm{d}z\right), \tag{3.14}$$

where $C_{ij}(x^k)$ denotes an arbitrary two-dimensional metric that depends only on the coordinates x^k.

At this point a somewhat lengthy but straightforward calculation shows that equation (2.28) for Φ is automatically solved.

We finally have to impose the Maxwell equation and Bianchi identity on $\mathcal{F}$. It is useful to define the complex variables

$$w = f - ig\,, \qquad F = \frac{2i}{\ell w}\,, \qquad \lambda = \exp\int F\,\mathrm{d}z\,, \tag{3.15}$$

in terms of which the combined Maxwell equation and Bianchi identity, $\mathrm{d}\left(\mathcal{F} + i{}^*\mathcal{F}\right) = 0$, reduce to

$$\bar\lambda\partial_z^3\lambda + \Delta_C F = 0\,, \tag{3.16}$$

where

$$\Delta_C \equiv \frac{1}{\sqrt{C}}\partial_i\left(\sqrt{C}C^{ij}\partial_j\right) = \lambda\bar\lambda\Delta_h \tag{3.17}$$

is the laplacian of the two-metric C_{ij} and Δ_h is the laplacian of h_{ij}. Equivalently, the equation reads

$$\partial_z^3\lambda + \lambda\,\Delta_h F = 0\,. \tag{3.18}$$

Hence, given an arbitrary two-metric C_{ij}, this complex equation determines the functions f and g.

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3.2 Integrability conditions and unbroken supersymmetry generators

To see if the solution is indeed invariant under some supersymmetry transformation, we have to construct the Killing spinor generating it. A necessary and sufficient condition for its existence (at least locally) is that the supercurvature vanishes.

First of all, as a consequence of the identities (2.11), the Killing spinor ϵ is subject to the constraint $\Gamma_{12}\epsilon = i\epsilon$, and this relation already breaks one half of the supersymmetries.

The t component of the Killing spinor equation yields $\partial_t\epsilon = 0$, hence ϵ is time-independent. Using (2.11), the other supercovariant derivatives simplify to

$$\begin{aligned}\hat{\nabla}_z\epsilon &= \left(\partial_z + \frac{1}{2N}W_+\partial_z W_- - \frac{i}{\ell}\mathcal{A}_z\right)\epsilon\,,\\ \hat{\nabla}_i\epsilon &= \left(\partial_i + \frac{1}{2N}W_+\partial_i W_- + \frac{i}{2}\check{\omega}_i^{12} - \frac{i}{\ell}\mathcal{B}_i\right)\epsilon\,,\end{aligned} \tag{3.19}$$

where we have defined $W_\pm = f \pm ig\Gamma_5$, and

$$\mathcal{B}_i = \mathcal{A}_i - f\omega_i\,. \tag{3.20}$$

The $W_\pm$ terms can be eliminated by decomposing the spinor ϵ into two chiral spinors $\eta_\pm$ defined by

$$\eta_+ = \frac{\epsilon_+}{\sqrt{w}}\,, \qquad \eta_- = \frac{\epsilon_-}{\sqrt{\bar{w}}}\,, \qquad \text{with} \qquad \epsilon_\pm = \frac{1}{2}\left(1 \pm \Gamma_5\right)\epsilon\,. \tag{3.21}$$

This leads to

$$\left(\partial_z - \frac{i}{\ell}\mathcal{A}_z\right)\eta_\pm = 0 \qquad \text{and} \qquad \left(\partial_i + \frac{i}{2}\check{\omega}_i^{12} - \frac{i}{\ell}\mathcal{B}_i\right)\eta_\pm = 0\,. \tag{3.22}$$

The identities (2.11) tell us that the left- and right-handed spinors η_- and η_+ are not independent, but related by $\eta_- = -i\Gamma_0\eta_+$. Together with $\eta_\pm = -i\Gamma_{12}\eta_\pm$, we deduce that generically we are left with only one quarter of the original supersymmetry.

The integrability conditions for (3.22) imply some additional constraints on the bosonic fields. From the vanishing of the (x, y) component of the supercurvature one gets

$$\check{\mathcal{R}}^{12} = \frac{2}{\ell}\check{\mathrm{d}}\mathcal{B}\,, \tag{3.23}$$

where $\check{\mathcal{R}}^{12} = \check{\mathrm{d}}\check{\omega}$ is the curvature form of the transverse two-manifold with metric h_{ij}. For the spin connection we have thus

$$\check{\omega} = \frac{2}{\ell}\mathcal{B} + \check{d}\chi\,, \tag{3.24}$$

with χ an arbitrary function of x, y and z. (3.24) is a "twisting" condition, since we can view the coupling to the gauge field $\mathcal{B}$ as effectively changing the spin of the supersymmetry parameter, which becomes a scalar [19].

The (z, i) component of the integrability conditions yields

$$\partial_i\left(\mathcal{A}_z + \frac{\ell}{2}\partial_z\chi\right) = 0\,, \tag{3.25}$$

and determines the function χ up to an arbitrary function $\psi(z)$,

$$\chi = -\frac{2}{\ell}\int \mathcal{A}_z\, \mathrm{d}z + \psi(z)\,. \tag{3.26}$$

Now, the Killing spinor equations reduce to

$$\left(\partial_\mu + \frac{i}{2}\partial_\mu\left(\chi - \psi\right)\right)\eta_\pm = 0\,, \tag{3.27}$$

and are solved by

$$\eta_\pm = e^{-\frac{i}{2}(\chi-\psi)}\eta^0_\pm = e^{\frac{i}{\ell}\int\mathcal{A}_z\,\mathrm{d}z}\eta^0_\pm\,, \tag{3.28}$$

where $\eta^0_\pm$ obey the constraints $\eta_- = -i\Gamma_0\eta_+$ and $\eta_\pm = -i\Gamma_{12}\eta_\pm$. These are satisfied by projecting an arbitrary spinor ϵ_0,

$$\eta_+ = \frac{1}{4}\left(1 - i\Gamma_{12}\right)\left(1+\Gamma_5\right)\epsilon_0\,, \qquad \eta_- = -\frac{i}{4}\Gamma_0\left(1 - i\Gamma_{12}\right)\left(1+\Gamma_5\right)\epsilon_0\,. \tag{3.29}$$

Finally, the Killing spinor reads

$$\epsilon = \frac{1}{4}e^{\frac{i}{\ell}\int\mathcal{A}_z\,\mathrm{d}z}\left(\sqrt{f-ig} - i\sqrt{f+ig}\,\Gamma_0\right)\left(1 - i\Gamma_{12}\right)\left(1+\Gamma_5\right)\epsilon_0\,. \tag{3.30}$$

We can now state our final result. Given an arbitrary two-metric $C_{ij}(x^k)$, one obtains the functions f and g by solving equation (3.16). Then, the shift vector ω is obtained as a solution of the equations (3.8) and (3.9). If in addition the integrability condition (3.24) is satisfied, the corresponding field configuration meets all necessary and sufficient conditions for the existence of the Killing spinor. In appendix E it is shown that these conditions, together with the Maxwell equation and Bianchi identity, imply in the timelike case that also the Einstein equations hold. In other words, these solutions are BPS states, preserving generically 1/4 of supersymmetry.

3.3 Reissner-Nordström-Taub-NUT-AdS solutions

As an example, we now show how to recover the supersymmetric Reissner-Nordström-Taub-NUT-AdS solutions obtained in [14]. Let C_{ij} be a metric of constant curvature k/l^2, where without loss of generality $k = 0, \pm 1$. An explicit form would be for instance[2]

$$C_{ij}dx^i dx^j = l^2(d\theta^2 + S_k^2(\theta)d\phi^2)\,, \tag{3.31}$$

where

$$S_k(\theta) = \begin{cases} 1\,, & k = 0 \\ \sin\theta\,, & k = 1 \\ \sinh\theta\,, & k = -1\,. \end{cases} \tag{3.32}$$

Let us further assume that the function F in eq. (3.15) does not depend on the coordinates θ, ϕ. We then obtain from (3.16)

$$\partial_z^3\lambda = 0\,, \tag{3.33}$$

[2] θ and ϕ are the coordinates previously termed x and y.

which leads to

$$w = \frac{2i}{l} \frac{az^2 + bz + c}{2az + b}, \tag{3.34}$$

where a, b, c are complex integration constants. Now we shift

$$z \to z - \frac{a\bar{b} + b\bar{a}}{4a\bar{a}}, \tag{3.35}$$

and define the real constants

$$\begin{aligned} n &= i\left(\frac{\bar{b}}{4\bar{a}} - \frac{b}{4a}\right), \\ Q &= \frac{i}{2l}\left(\frac{c}{a} - \frac{\bar{c}}{\bar{a}}\right), \\ P &= -\frac{1}{2l}\left(2n^2 + \frac{c}{a} + \frac{\bar{c}}{\bar{a}}\right). \end{aligned}$$

This yields

$$f = -\frac{n}{l} + \frac{Qz - nP}{z^2 + n^2}, \qquad g = -\frac{z}{l} + \frac{Pz + nQ}{z^2 + n^2}, \tag{3.36}$$

and

$$-Nl^2 = \frac{z^4 + 2z^2(n^2 - lP) - 4nlQz + n^2(n^2 + 2lP) + l^2(Q^2 + P^2)}{z^2 + n^2}. \tag{3.37}$$

The shift vector ω_i can now be determined from (3.8) and (3.9), with the result $\omega_\theta = 0$ and

$$\omega_\phi = \begin{cases} -2cn\theta\,, & k = 0 \\ 2cn\cos\theta\,, & k = 1 \\ -2cn\cosh\theta\,, & k = -1\,, \end{cases} \tag{3.38}$$

where c denotes an integration constant that can be set equal to one by rescaling $z \to \gamma z$, $t \to t/\gamma$, $n \to \gamma n$, $Q \to \gamma^2 Q$, $P \to \gamma^2 P$, with $\gamma^{-2} = c$. The metric finally reads

$$ds^2 = N(dt + \omega_\phi d\phi)^2 - \frac{dz^2}{N} + (z^2 + n^2)l^{-2}C_{ij}dx^i dx^j\,, \tag{3.39}$$

with C_{ij}, N and ω_ϕ given in eqns. (3.31), (3.37) and (3.38) respectively. The electromagnetic field strength is easily obtained from (3.5). One checks that the final solution belongs to the Reissner-Nordström-Taub-NUT-AdS class of solutions [14], with arbitrary nut charge n and electric charge Q, whereas the magnetic charge P and mass parameter M are given by

$$P = -\frac{kl^2 + 4n^2}{2l}, \qquad M = \frac{2nQ}{l}. \tag{3.40}$$

These are exactly the conditions on P and M found in [14], under which the RN-TN-AdS solutions preserve one quarter of the supersymmetry.[3] Note that the integrability conditions (3.23) are already satisfied by our solution (3.39), (3.40), because they yield exactly the quantization of the magnetic charge given in (3.40).

It would be interesting to recover also the rotating BPS black holes studied in [12].

[3]The other sign for P given in [14] can be obtained by replacing $\phi \to -\phi$, $n \to -n$.

3.4 New BPS solutions of Petrov type I

All previously known timelike supersymmetric solutions of $N = 2$, $D = 4$ gauged supergravity were included in the Plebański-Demiański class of solutions, which are of Petrov type D (or O). To show that new supersymmetric geometries arise from the most general solution, we will now explicitly extract a BPS solution of Petrov type I.

To simplify the equations, we restrict ourselves to $g = 0$ solutions, and define a new function $H \equiv 1/f$. The equations in subsection 3.1 can then be easily solved; it turns out that H can be chosen to be an arbitrary function of x and y such that $\Delta H \neq 0$, and the solution reads

$$\mathrm{d}s^2 = -\frac{1}{H^2}\left(\mathrm{d}t + \omega_i\,\mathrm{d}x^i\right)^2 + H^2\,\mathrm{d}z^2 + \frac{\ell^2\Delta H}{4H}\left(\mathrm{d}x^2 + \mathrm{d}y^2\right), \tag{3.41}$$

where ω_i is any solution of the curl equation

$$\partial_x\omega_y - \partial_y\omega_x = \ell\Delta H\,, \tag{3.42}$$

and the corresponding electromagnetic field reads

$$\mathcal{F} = -\partial_i\left(\frac{1}{H}\right)\left(\mathrm{d}t + \omega_j\,\mathrm{d}x^j\right)\wedge\mathrm{d}x^i + \frac{\ell\Delta H}{4H}\,\mathrm{d}x\wedge\mathrm{d}y\,. \tag{3.43}$$

The integrability condition (3.24) reduces then to the partial differential equation

$$\Delta\ln\Delta H = 6\frac{\Delta H}{H} - 3\frac{(\nabla H)^2}{H^2}\,. \tag{3.44}$$

It is easy to find a particular solution of these equations, given by

$$H = \frac{x^2}{\ell^2} \tag{3.45}$$

and

$$\omega = \frac{2x}{\ell}\,\mathrm{d}y\,. \tag{3.46}$$

This yields the BPS solution

$$\mathrm{d}s^2 = -\frac{\ell^4}{x^4}\left(\mathrm{d}t + \frac{2x}{\ell}\,\mathrm{d}y\right)^2 + \frac{x^4}{\ell^4}\,\mathrm{d}z^2 + \frac{\ell^2}{2x^2}\left(\mathrm{d}x^2 + \mathrm{d}y^2\right), \tag{3.47}$$

$$\mathcal{F} = \frac{2\ell^2}{x^3}\,\mathrm{d}t\wedge\mathrm{d}x - \frac{7\ell}{2x^2}\,\mathrm{d}x\wedge\mathrm{d}y\,. \tag{3.48}$$

This metric has four Killing vectors,

$$T = \frac{\partial}{\partial t}\,,\qquad Y = \frac{\partial}{\partial y}\,,\qquad Z = \frac{\partial}{\partial z}\,,\qquad K = t\frac{\partial}{\partial t} - z\frac{\partial}{\partial z} + \frac{x}{2}\frac{\partial}{\partial x} + \frac{y}{2}\frac{\partial}{\partial y}\,, \tag{3.49}$$

acting transitively on the whole spacetime. Hence the solution represents a homogeneous, stationary and geodesically complete BPS spacetime endowed with a nonnull electromagnetic field. A computation of the Weyl scalars of this metric shows that its Petrov type is I. In the context of Einstein-Maxwell theory, this solution was obtained in [20].

4. The lightlike case

4.1 Construction of the bosonic fields

In the case when the Killing vector constructed from the Killing spinor is lightlike, the algebraic constraints (2.14)–(2.18) simplify to

$$V^2 = A^2 = f = g = \Phi^2 = 0\,, \tag{4.1}$$

$$V^\mu \Phi_{\mu\nu} = A^\mu \Phi_{\mu\nu} = 0\,, \qquad A \wedge \Phi = V \wedge \Phi = 0\,, \qquad \Phi \wedge \Phi = 0\,, \tag{4.2}$$

$$V \wedge A = 0\,, \qquad \Phi_{(\mu}{}^{\rho} \varepsilon_{\nu)\rho\alpha\beta} = 0\,. \tag{4.3}$$

The differential constraints read

$$\mathcal{F}_{\mu\nu} V^\nu = 0\,, \tag{4.4}$$

$$\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma} V^\nu \mathcal{F}^{\rho\sigma} = -\frac{1}{\ell} A_\mu\,, \tag{4.5}$$

$$\nabla_\mu V_\nu = \frac{1}{\ell}\Phi_{\mu\nu}\,, \tag{4.6}$$

$$\nabla_\mu A_\nu = -\mathcal{F}_{(\mu}{}^{\rho}\varepsilon_{\nu)\rho\alpha\beta}\Phi^{\alpha\beta} + \frac{1}{4} g_{\mu\nu}\varepsilon_{\rho\sigma\alpha\beta}\mathcal{F}^{\rho\sigma}\Phi^{\alpha\beta}\,, \tag{4.7}$$

$$\nabla_\mu \Phi_{\alpha\beta} = \frac{2}{\ell} g_{\mu[\alpha} V_{\beta]} + 2\mathcal{F}_{[\alpha}{}^{\sigma}\varepsilon_{\beta]\sigma\mu\nu}A^\nu + \mathcal{F}_\mu{}^{\sigma}\varepsilon_{\sigma\alpha\beta\nu}A^\nu + g_{\mu[\alpha}\varepsilon_{\beta]\nu\rho\sigma}A^\nu \mathcal{F}^{\rho\sigma}\,. \tag{4.8}$$

From (4.2) and (4.6) we get $V \wedge \mathrm{d}V = 0$, hence V is hypersurface orthogonal, and there exist two functions H and u such that $V = H^{-1}\,\mathrm{d}u$. Furthermore one has $V^\mu \nabla_\mu V_\nu = \frac{1}{\ell} V^\mu \Phi_{\mu\nu} = 0$, so that V is tangent to an affinely parametrized congruence of geodesics in the surfaces of constant u. Let v be the affine parameter, $V = \partial/\partial v$. Now use u and v as coordinates and choose coordinates y^m, $m = 1, 2$ for the transverse space. The most general metric reads

$$\mathrm{d}s^2 = \frac{1}{H}\left(\mathcal{G}\,\mathrm{d}u^2 + 2\,\mathrm{d}u\mathrm{d}v\right) + H^{2\alpha}\gamma_{mn}\,\mathrm{d}y^m \mathrm{d}y^n\,, \tag{4.9}$$

with α a real parameter to be chosen for convenience. As V is a Killing vector,H, $\mathcal{G}$ and γ_{mn} are functions of (u, y^m) only. Let us introduce a null basis (e^+, e^-, e^i) for the tangent space, given by

$$e^+ = V = \frac{1}{H}\,\mathrm{d}u\,, \qquad e^- = \mathrm{d}v + \frac{1}{2}\mathcal{G}\,\mathrm{d}u\,, \qquad e^i = H^\alpha\,\check{e}^i\,, \tag{4.10}$$

with $\check{e}^i = \check{e}^i{}_m\,\mathrm{d}y^m$, $\delta_{ij}\check{e}^i\check{e}^j = \gamma_{mn}\mathrm{d}y^m\mathrm{d}y^n$. From (4.6) we obtain then $\Phi = -\dfrac{\ell}{2H}\,\mathrm{d}H \wedge e^+$. The equation $V \wedge A = 0$ yields $A = k(u, y^m)V = k(u, y^m)e^+$. The function $k(u, y^m)$ can then be determined from the condition $\mathrm{d}A = 0$ and the definition of e^+. In this way one obtains $k(u, y^m) = \kappa(u)H(u, y^m)$, where $\kappa(u)$ is a function of u only. By a reparametrization $u = u(u')$ of the coordinate u, it is possible to set κ to a constant. The only two inequivalent solutions are $\kappa = 0$ (hence $A = 0$) and $\kappa = 1$:

$$A = \kappa H(u, y^m)\,e^+\,, \qquad \kappa = 0, 1\,. \tag{4.11}$$

The transverse manifold is two-dimensional, hence conformally flat. It is then possible to choose coordinates $x^i = x^i(u, y^m)$, $i = 1, 2$, such that $\gamma_{mn}\, \mathrm{d}y^m \mathrm{d}y^n = \Omega^2(u, x^1, x^2)[(\mathrm{d}x^1)^2 + (\mathrm{d}x^2)^2]$.

Note that this coordinate transformation introduces mixed terms $\mathrm{d}u\mathrm{d}x^i$. In the coordinates (u, v, x^i), the metric reads

$$\mathrm{d}s^2 = \frac{1}{H}\left(\mathcal{G}\ \mathrm{d}u^2 + 2\ \mathrm{d}u\mathrm{d}v\right) + H^{2\alpha}\Omega^2\left(\mathrm{d}x^i + a^i\,\mathrm{d}u\right)\left(\mathrm{d}x^i + a^i\,\mathrm{d}u\right), \tag{4.12}$$

where H, $\mathcal{G}$, Ω and a^i are functions of (u, x^i). The vierbein is now given by

$$e^+ = V = \frac{1}{H}\,\mathrm{d}u\,, \qquad e^- = \mathrm{d}v + \frac{1}{2}\mathcal{G}\,\mathrm{d}u\,, \qquad e^i = H^\alpha\Omega\left(\mathrm{d}x^i + a^i\,\mathrm{d}u\right). \tag{4.13}$$

Finally we choose an orientation such that $\varepsilon_{+-12} = \eta$ with $\eta^2 = 1$.

In order to determine the Maxwell field, we use $V^\mu\mathcal{F}_{\mu\nu} = 0$, which yields

$$\mathcal{F} = \mathcal{F}_{+i}e^+ \wedge e^i + \frac{1}{2}\mathcal{F}_{ij}e^i \wedge e^j\,. \tag{4.14}$$

The dual field is

$${}^*\mathcal{F} = \eta\mathcal{F}_{12}\,e^+ \wedge e^- - \eta\epsilon_{ij}\mathcal{F}_{+j}\,e^+ \wedge e^i\,, \tag{4.15}$$

where we have defined $\epsilon_{12} = -\epsilon_{21} = 1$. From (4.5) and (4.11) one gets then $\mathcal{F}_{12} = -\frac{\eta\kappa}{\ell}H$, and finally

$$\begin{aligned}\mathcal{F} &= \mathcal{F}_{+i}e^+ \wedge e^i - \frac{\eta\kappa}{\ell}H\,e^1 \wedge e^2\,,\\ {}^*\mathcal{F} &= -\frac{\kappa}{\ell}H\,e^+ \wedge e^- - \eta\epsilon_{ij}\mathcal{F}_{+j}\,e^+ \wedge e^i\,.\end{aligned} \tag{4.16}$$

The Bianchi identity $\mathrm{d}F = 0$ and the Maxwell equation $\mathrm{d}^*\mathcal{F} = 0$ imply

$$\partial_i\left(\Omega H^{\alpha-1}\epsilon_{ij}\mathcal{F}_{+j}\right) = -\frac{\eta\kappa}{\ell}\left[\partial_u\left(\Omega^2H^{2\alpha+1}\right) - \partial_i\left(\Omega^2H^{2\alpha+1}a^j\right)\right] \tag{4.17}$$

and

$$\partial_i\left(\Omega H^{\alpha-1}\mathcal{F}_{+i}\right) = 0 \tag{4.18}$$

respectively. The general solution of (4.18) is

$$\mathcal{F}_{+i} = \Omega^{-1}H^{-\alpha+1}\epsilon_{ij}\partial_j\psi\,, \tag{4.19}$$

where the function $\psi(u, x^i)$ is determined by the Bianchi identity (4.17), which yields

$$\Delta\psi = \frac{\eta\kappa}{\ell}\left[\partial_u\left(\Omega^2H^{2\alpha+1}\right) - \partial_i\left(\Omega^2H^{2\alpha+1}a^i\right)\right], \tag{4.20}$$

where $\Delta \equiv \partial_1^2 + \partial_2^2$ is the flat transverse space laplacian.

We come now to the differential equation (4.7) for A. As we have

$$\varepsilon_{\rho\sigma\alpha\beta}\mathcal{F}^{\rho\sigma}\Phi^{\alpha\beta} = 0\,, \tag{4.21}$$

it simplifies to

$$\nabla_\mu A_\nu = -\mathcal{F}_{(\mu}{}^\rho\varepsilon_{\nu)\rho\alpha\beta}\Phi^{\alpha\beta}\,. \tag{4.22}$$

Using (4.11) and (4.6) yields

$$\nabla_\mu A_\nu = \kappa V_\nu \partial_\mu H + \frac{\kappa}{\ell} H \Phi_{\mu\nu}\,. \tag{4.23}$$

Comparing this with (4.22), one obtains in the orthonormal frame

$$\kappa V_a \partial_\mu H = \frac{\kappa}{\ell} H e^b{}_\mu \mathcal{F}_{(a}{}^c\, ({}^*\Phi)_{b)c}\,. \tag{4.24}$$

The only non trivial statement comes from the components $a = +$ and $\mu = u$,

$$\kappa \partial_u H = \left(\kappa a^k + \frac{\eta \ell}{\Omega^2 H^{2\alpha+1}} \partial_k \psi \right) \partial_k H\,. \tag{4.25}$$

In order to solve the differential equation (4.8) for Φ, we use (4.6) to get

$$\nabla_\mu \Phi_{\alpha\beta} = \ell \nabla_\mu \nabla_\alpha V_\beta\,. \tag{4.26}$$

Since V is a Killing vector, we have

$$\nabla_\mu \nabla_\nu V_\rho = -R_{\nu\rho\mu}{}^\sigma V_\sigma\,. \tag{4.27}$$

In the orthonormal frame, (4.8) becomes then

$$-\ell R_{abc}{}^+ = \frac{2}{\ell} \eta_{c[a} V_{b]} + 2\mathcal{F}_{[a}{}^d \varepsilon_{b]dce} A^e + \mathcal{F}_c{}^d \varepsilon_{dabe} A^e + \eta_{c[a} \varepsilon_{b]def} A^d \mathcal{F}^{ef}\,. \tag{4.28}$$

The independent components of this equation are

$$\begin{aligned}
R_{+-+-} &= \frac{1}{\ell^2}\left(1 - \kappa H^2\right), && (4.29)\\
R_{+-+i} &= \frac{2\eta\kappa}{\ell \Omega H^{\alpha-1}} \partial_i \psi\,, && (4.30)\\
R_{+-ij} &= \frac{2\kappa}{\ell^2} H^2 \epsilon_{ij}\,, && (4.31)\\
R_{+i-j} &= -\frac{1}{\ell^2}\left(1 + \kappa H^2\right) \delta_{ij}\,, && (4.32)\\
R_{+--i} &= R_{-i-j} = R_{-ijk} = 0\,. && (4.33)
\end{aligned}$$

The equations (4.33) are automatically satisfied by the Riemann tensor (D.2), (D.3) of the metric (4.12).

From (4.31) we have $\frac{2\kappa}{\ell^2} H^2 \epsilon_{ij} = 0$, and thus $\kappa = 0$, because $H = 0$ would lead to a degenerate metric. The remaining equations then simplify to

$$\begin{aligned}
\frac{\partial_i H \partial_i H}{4\Omega^2 H^{2\alpha+2}} &= \frac{1}{\ell^2}\,, && (4.34)\\
a^k \partial_k \partial_i (\ln H) - \partial_u \partial_i (\ln H) + \partial_i (\ln H)\, \partial_u (\ln \Omega H^\alpha) + \partial_{[i} a_{j]} \partial_j (\ln H) &= 0\,, && (4.35)\\
\frac{1}{2\Omega^2 H^{2\alpha}} \Bigg[\frac{1}{H} \partial_i \partial_j H - 2\partial_{(i} \ln H \partial_{j)} \ln \Omega - \left(2\alpha + \frac{3}{2}\right) \partial_i \ln H \partial_j \ln H + &\\
+ \delta_{ij} \partial_k (\ln H)\, \partial_k (\ln \Omega H^\alpha) \Bigg] &= -\frac{1}{\ell^2} \delta_{ij}\,, && (4.36)
\end{aligned}$$

in addition to $\Delta\psi = 0$ following from (4.20) and $\partial_k \psi \partial_k H = 0$ from (4.25).

The Einstein equations are almost automatically satisfied if the fields solve the previous conditions for the existence of a Killing spinor. The only component that has to be verified explicitly is the $(+,+)$ one (cf. appendix E and ref. [4]). The Einstein equations read

$$G_{\mu\nu} - \frac{3}{\ell^2} g_{\mu\nu} = 2\left(\mathcal{F}_\mu{}^\rho \mathcal{F}_{\nu\rho} - \frac{1}{4}\mathcal{F}^2 g_{\mu\nu}\right), \tag{4.37}$$

whose $(+,+)$ component yields

$$R_{++} = 2\mathcal{F}_{+i}\mathcal{F}_{+i} = \frac{2\partial_i\psi\partial_i\psi}{\Omega^2 H^{2\alpha-2}}, \tag{4.38}$$

where we have substituted $\mathcal{F}_{+i}$ from (4.19). The component R_{++} of the Ricci tensor is given in (D.4).

We have thus to solve the equations (4.34)–(4.36), the $(+,+)$ component of the Einstein equation, and the equations for ψ.

The trace of eqn. (4.36) yields

$$\frac{1}{H}\Delta H - \frac{3}{2}\frac{(\nabla H)^2}{H^2} = -\frac{4}{\ell^2}\Omega^2 H^{2\alpha}. \tag{4.39}$$

Using eqn. (4.34) to eliminate Ω, one obtains

$$\Delta H - \frac{1}{2H}(\nabla H)^2 = 0. \tag{4.40}$$

This equation is equivalent to $\Delta H^{1/2} = 0$, hence $H^{1/2}$ solves the flat Laplace equation on the (x,y) plane. We introduce now complex coordinates $\zeta = x^1 + ix^2$, $\bar\zeta = x^1 - ix^2$ and the holomorphic and anti-holomorphic derivative operators $\partial = \frac{1}{2}(\partial_1 - i\partial_2)$, $\bar\partial = \frac{1}{2}(\partial_1 + i\partial_2)$. The equation for H now reads $\partial\bar\partial H^{1/2} = 0$, and the general solution is the sum of a holomorphic and an anti-holomorphic function,

$$H^{1/2} = h(u,\zeta) + \tilde h(u,\bar\zeta). \tag{4.41}$$

From the reality of H we have

$$\tilde h(u,\zeta) = \overline{h(u,\zeta)} = \bar h(u,\bar\zeta). \tag{4.42}$$

Finally,

$$H(u,\zeta,\bar\zeta) = \left[h(u,\zeta) + \bar h(u,\zeta)\right]^2, \tag{4.43}$$

where $h(u,\zeta)$ is an arbitrary function.

The conformal factor Ω can now be determined using again eqn. (4.34), with the result

$$\Omega(u,\zeta,\bar\zeta) = \frac{2\ell}{\left(h+\bar h\right)^\alpha}\left(\partial h \bar\partial \bar h\right)^{1/2}. \tag{4.44}$$

The component $(1,2)$ of Eqn. (4.34) reduces to

$$(\alpha+1)\left[(\partial h)^2 - \left(\bar\partial \bar h\right)^2\right] = 0. \tag{4.45}$$

We can, with no loss of generality, set $\alpha = -1$, so that (4.45) imposes no further restriction on the arbitrary function h.

Using the expressions (4.43) for H and (4.44) for Ω, the metric reads

$$\mathrm{d}s^2 = \frac{1}{(h+\bar{h})^2}\left(\mathcal{G}\,\mathrm{d}u^2 + 2\,\mathrm{d}u\mathrm{d}v + 4\ell^2\partial h\bar{\partial}\bar{h}\;|\mathrm{d}\zeta + a\,\mathrm{d}u|^2\right). \tag{4.46}$$

We can define new coordinates $\xi = 2\ell h(\zeta,u)$ and $\bar{\xi} = 2\ell\bar{h}(\zeta,u)$, to obtain

$$\mathrm{d}s^2 = \frac{4\ell^2}{(\xi+\bar{\xi})^2}\left(\mathcal{G}\,\mathrm{d}u^2 + 2\,\mathrm{d}u\mathrm{d}v + |\mathrm{d}\xi + b\,\mathrm{d}u|^2\right), \tag{4.47}$$

where the function b is defined by $b = -2\ell(\partial_u - a^k\partial_k)h$. This is exactly a metric of the original form, with $h(\zeta,u) = \zeta/2\ell$ and $a = b$. Thus, without loss of generality, we can restrict to solutions with

$$H = \left(\frac{\zeta+\bar{\zeta}}{2\ell}\right)^2, \qquad \Omega = \frac{\zeta+\bar{\zeta}}{2\ell}. \tag{4.48}$$

In the coordinates $x^k = (x,y)$ (recall $\zeta = x+iy$), the metric reads

$$\mathrm{d}s^2 = \frac{\ell^2}{x^2}\left(\mathcal{G}\,\mathrm{d}u^2 + 2\,\mathrm{d}u\mathrm{d}v + \left(\mathrm{d}x^k + a^k\,\mathrm{d}u\right)\left(\mathrm{d}x^k + a^k\,\mathrm{d}u\right)\right). \tag{4.49}$$

In order to determine a^k we have to solve Eqn. (4.35), which simplifies to

$$a^x = 0\,, \qquad \partial_x a^y = 0\,. \tag{4.50}$$

Hence $a^y = a(u,y)$ is an arbitrary function of u and y.

The electromagnetic field is determined by the scalar field ψ, restricted by $\Delta\psi = 0$, $\nabla\psi\cdot\nabla H = 0$. The second equation implies $\partial_x\psi = 0$, hence $\psi = \psi(u,y)$. Then the first one imposes $\partial_y^2\psi = 0$, which is satisfied by

$$\psi = y\varphi'(u) + \psi_0(u)\,, \tag{4.51}$$

where φ and ψ_0 are two arbitrary functions of u. The resulting electromagnetic field is $\mathcal{F} = \varphi'(u)\,\mathrm{d}u\wedge\mathrm{d}x$, and does not depend on ψ_0, hence we can take $\psi_0 = 0$ in the following.

Now the metric reads

$$\mathrm{d}s^2 = \frac{\ell^2}{x^2}\left(\mathcal{G}(u,x,y)\,\mathrm{d}u^2 + 2\,\mathrm{d}u\mathrm{d}v + \mathrm{d}x^2 + (\mathrm{d}y + a(u,y)\,\mathrm{d}u)^2\right), \tag{4.52}$$

where the function $\mathcal{G}$ must satisfy the $(++)$ component of Einstein's equations,

$$\Delta\mathcal{G} - \frac{2}{x}\partial_x\mathcal{G} = -\frac{4x^2}{\ell^2}\left(\varphi'\right)^2 + 2\partial_y\partial_u a - 2\partial_y\left(a\partial_y a\right). \tag{4.53}$$

This equation reduces to the Siklos equation [21] when $\varphi = a = 0$.

We can eliminate $a(u,y)$ by shifting $v \mapsto v + g(u,x,y)$, with g an arbitrary function of its arguments. The metric remains of the same form, and the functions $\mathcal{G}$ and the vector a^k transform according to

$$\begin{aligned} \mathcal{G} &\mapsto \mathcal{G}' = \mathcal{G} + 2g_{,u} - 2a\cdot\nabla g - (\nabla g)^2\,, \\ a^k &\mapsto a'^k = a^k + \partial^k g\,. \end{aligned} \tag{4.54}$$

Thus, taking $g(u,x,y) = -\int a(u,y)\,\mathrm{d}y$ in (4.52), we can eliminate the vector a^k and, up to diffeomorphisms, the most general solution has $a = 0$.

4.2 Integrability conditions and unbroken supersymmetry generators

We finally come to the unbroken supersymmetry generators. First of all, we have $f = g = 0$, hence $V^\mu \Gamma_\mu \epsilon = i(f + ig\Gamma_5)\epsilon = 0$, and given $V = e^+$ it follows that

$$\Gamma^+ \epsilon = 0\,, \tag{4.55}$$

so half of the supersymmetries is broken. The v component of the Killing spinor equation yields

$$\frac{\partial \epsilon}{\partial v} = 0\,, \tag{4.56}$$

so that ϵ is independent of v. The y component reads

$$\left[\partial_y + \frac{1}{2x}\Gamma_2\left(1 - \Gamma_1\right)\right]\epsilon = 0\,, \tag{4.57}$$

and is solved by taking

$$\frac{\partial \epsilon}{\partial y} = 0\,, \qquad \left(1 - \Gamma_1\right)\epsilon = 0\,. \tag{4.58}$$

For such a spinor, the remaining equations (u and x components) reduce to

$$\left(\partial_x - \frac{i}{\ell}\varphi + \frac{1}{2x}\Gamma_1\right)\epsilon = 0\,, \qquad \left(\partial_u - \frac{ix}{\ell}\varphi'\Gamma_1\right)\epsilon = 0\,. \tag{4.59}$$

This system admits the solution

$$\epsilon = \sqrt{\frac{\ell}{x}}\exp\left(\frac{ix}{\ell}\varphi(u)\right)\epsilon_0\,, \tag{4.60}$$

where ϵ_0 is an arbitrary constant spinor satisfying

$$\Gamma_- \epsilon_0 = 0\,, \qquad \left(1 - \Gamma_1\right)\epsilon_0 = 0\,. \tag{4.61}$$

Hence the lightlike supersymmetric solutions are one quarter BPS states.

4.3 The general lightlike BPS solution

In conclusion, the general lightlike BPS solution is an electrovac travelling wave given by an arbitrary function $\varphi(u)$ and a solution $\mathcal{G}(u, x, y)$ of the equation

$$\Delta\mathcal{G} - \frac{2}{x}\partial_x\mathcal{G} = -\frac{4x^2}{\ell^2}\left(\varphi'\right)^2\,. \tag{4.62}$$

The metric reads

$$\mathrm{d}s^2 = \frac{\ell^2}{x^2}\left[\mathcal{G}(u, x, y)\,\mathrm{d}u^2 + 2\,\mathrm{d}u\mathrm{d}v + \mathrm{d}x^2 + \mathrm{d}y^2\right], \tag{4.63}$$

and the null electromagnetic field is given by

$$\mathcal{F} = \mathrm{d}\mathcal{A} = \varphi'(u)\,\mathrm{d}u \wedge \mathrm{d}x\,, \qquad \mathcal{A} = \varphi(u)\,\mathrm{d}x\,. \tag{4.64}$$

This family of solutions has a Virasoro symmetry with non-zero central charge as in the pure gravitational case [22], corresponding to the reparameterization freedom in the coordinate u, $u = f(u')$. Defining a new complex coordinate ξ by

$$x + iy = \ell\,\frac{\xi - \ell}{\xi + \ell}\,, \tag{4.65}$$

and performing the change of coordinates, we see that these solutions form the $(IV)_0$ family of metrics found in [23], describing exact gravitational and electromagnetic waves of arbitrary profile propagating in AdS space along the direction v [24]. The special case

$$\mathcal{G} = \frac{P^2}{\ell^4}x^3 - \frac{P_e^2}{\ell^6}x^3 \tag{4.66}$$

is a charged Kaigorodov space, obtained in [25] as the ultrarelativistic limit of a boosted non-extremal charged domain wall.

5. Maximal supersymmetry

In order to get a maximally supersymmetric solution, we require that the integrability condition (E.1) following from the Killing spinor equation imposes no algebraic constraints on the Killing spinor. This means that the terms which are zeroth, first, second, third and fourth order in the gamma-matrices must vanish independently. From the zeroth order term we immediately obtain $\mathcal{F} = 0$.[4] Using this, the integrability condition simplifies to

$$R_{\alpha\beta\nu\mu} = \frac{1}{\ell^2}(g_{\alpha\nu}g_{\beta\mu} - g_{\alpha\mu}g_{\beta\nu})\,, \tag{5.1}$$

so that the spacetime has constant curvature. We conclude that the only maximally supersymmetric geometry is given by AdS_4 with vanishing gauge field. This is analogous to the five-dimensional case [5].

6. Final remarks

We conclude this paper by pointing out some possible extensions of the work presented here. First of all, it would be interesting to uplift e.g. the new lightlike solutions to eleven dimensions and to study their M-theory interpretation.

Furthermore, one could refine our classification in the sense of finding the additional restrictions on the geometries in order that they preserve more than one supersymmetry, and to see whether e.g. 3/4 supersymmetric solutions are possible.

Although electromagnetic duality invariance is broken in the gauged theory due to the minimal coupling of the gravitini to the graviphoton, a generalized duality invariance was discovered in the supersymmetric subclass of the Plebański-Demiański solution, which rotates also the mass parameter into the nut charge and vice-versa [14]. It would be interesting to see whether this duality can be understood within our formalism.

[4] Note that this zeroth order term comes with prefactor $1/\ell$, so that it is absent in the ungauged case, where maximally supersymmetric solutions with nonvanishing gauge field exist.

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Finally, one could consider matter-coupled gauged supergravity, where a large class of supersymmetric black holes is known [26, 27], and see whether a classification of BPS solutions is still feasible. Work in these directions is in progress.

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A. Conventions

Throughout this paper, the conventions are as follows: $a, b, \ldots$ refer to $D = 4$ tangent space indices, and $\mu, \nu, \ldots$ refer to $D = 4$ world indices. The signature is $(-,+,+,+)$, $\varepsilon_{0123} = +1$.

The gamma matrices are defined to satisfy the four-dimensional Clifford algebra $\{\Gamma_a, \Gamma_b\} = 2\eta_{ab}$, and the parity matrix is $\Gamma_5 = i\Gamma_{0123}$. We antisymmetrize with unit weight, i.e. $\Gamma_{ab} \equiv \Gamma_{[a}\Gamma_{b]} \equiv \frac{1}{2}[\Gamma_a, \Gamma_b]$ etc. The Dirac conjugate is defined by $\bar{\psi} = i\psi^\dagger\Gamma^0$.

Finally, for two-dimensional submanifolds, we use $i, j, \ldots$ as $D = 2$ indices. The indices can take the value 1, 2 and $\epsilon_{12} = +1$. The geometric objects associated to the bidimensional metric h_{ij} are surmounted by a check sign; hence $\check{\omega}^{12}{}_i$ is the spin connection, and the curvature tensor reads $\check{\mathcal{R}}^{12} = \check{\mathrm{d}}\check{\omega}^{12}$, where $\check{\mathrm{d}} = \mathrm{d}x\,\partial_x + \mathrm{d}y\,\partial_y$ is the two-dimensional exterior derivative. We denote by

$$\Delta_h \equiv \frac{1}{\sqrt{h}}\partial_i\left(\sqrt{h}h^{ij}\partial_j\right) \tag{A.1}$$

the laplacian associated to h_{ij}, and with Δ the usual flat bidimensional laplacian.

B. Decomposition of a two-form with respect to a vector K

Given a two-form F and a vector K, we can decompose F in an electric and a magnetic part, defined by

$$E = -i_K F = {}^*\left(K \wedge {}^*F\right), \qquad B = i_K{}^*F = {}^*\left(K \wedge F\right). \tag{B.1}$$

Then the two-form reads

$$F = -\frac{1}{K^2}\left(K \wedge E + {}^*\left(K \wedge B\right)\right). \tag{B.2}$$

If K is a Killing vector, it follows from the definition that $\mathcal{L}_K E = \mathcal{L}_K B = 0$.

C. Useful relations for a spinor ϵ

$$
\begin{aligned}
\bar\epsilon\Gamma_\mu\Gamma_\nu\epsilon &= fg_{\mu\nu} - i\Phi_{\mu\nu}\,, \qquad \bar\epsilon\Gamma_5\Gamma_\mu\Gamma_\nu\epsilon = -igg_{\mu\nu} + \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\Phi^{\rho\sigma}\,,\\
\bar\epsilon\Gamma_{\mu\nu}\Gamma_\rho\epsilon &= -\varepsilon_{\mu\nu\rho\sigma}A^\sigma - 2iV_{[\mu}g_{\nu]\rho}\,, \qquad \bar\epsilon\Gamma_\mu\Gamma_{\nu\rho}\epsilon = -\varepsilon_{\mu\nu\rho\sigma}A^\sigma - 2ig_{\mu[\nu}V_{\rho]}\,,\\
\bar\epsilon\Gamma_5\Gamma_{\mu\nu}\Gamma_\rho\epsilon &= -\varepsilon_{\mu\nu\rho\sigma}V^\sigma - 2iA_{[\mu}g_{\nu]\rho}\,,\\
\bar\epsilon\Gamma_5\Gamma_\mu\Gamma_{\nu\rho}\epsilon &= -\varepsilon_{\mu\nu\rho\sigma}V^\sigma - 2ig_{\mu[\nu}A_{\rho]}\,,\\
\bar\epsilon\Gamma_{\mu\nu}\Gamma_{\rho\sigma}\epsilon &= -g\varepsilon_{\mu\nu\rho\sigma} + 2i\Phi_{\mu[\rho}g_{\sigma]\nu} - 2ig_{\mu[\rho}\Phi_{\sigma]\nu} - 2fg_{\mu[\rho}g_{\sigma]\nu}\,,\\
\bar\epsilon\Gamma_5\Gamma_{\mu\nu}\Gamma_{\rho\sigma}\epsilon &= -if\varepsilon_{\mu\nu\rho\sigma} + \varepsilon_{\rho[\mu}{}^{\alpha\beta}g_{\nu]\sigma}\Phi_{\alpha\beta} - \varepsilon_{\sigma[\mu}{}^{\alpha\beta}g_{\nu]\rho}\Phi_{\alpha\beta} + 2igg_{\mu[\rho}g_{\sigma]\nu}\,,\\
\bar\epsilon\Gamma_\mu\Gamma_{\rho\sigma}\Gamma_\nu\epsilon &= -g\varepsilon_{\mu\nu\rho\sigma} - 2i\Phi_{\mu[\rho}g_{\sigma]\nu} - 2ig_{\mu[\rho}\Phi_{\sigma]\nu} - ig_{\mu\nu}\Phi_{\rho\sigma} + 2fg_{\mu[\rho}g_{\sigma]\nu}\,,\\
\bar\epsilon\Gamma_5\Gamma_\mu\Gamma_{\rho\sigma}\Gamma_\nu\epsilon &= -if\varepsilon_{\mu\nu\rho\sigma} - 2igg_{\mu[\rho}g_{\sigma]\nu} + \varepsilon_{\mu[\rho}{}^{\alpha\beta}g_{\sigma]\nu}\Phi_{\alpha\beta} +\\
&\quad + g_{\mu[\rho}\varepsilon_{\sigma]\nu}{}^{\alpha\beta}\Phi_{\alpha\beta} + \frac{1}{2}g_{\mu\nu}\varepsilon_{\rho\sigma\alpha\beta}\Phi^{\alpha\beta}\,.
\end{aligned}
\tag{C.1}
$$

D. Geometry of the lightlike solution

Defining $\tilde\nabla \equiv \partial_u - a^k\partial_k$, the spin connection and the components of the Riemann and Ricci tensors needed to solve the lightlike case read as follows.

Spin connection:

$$
\begin{aligned}
\omega_{+-} &= -\frac{\partial_i H}{2\Omega H^{\alpha+1}}\,e^i\,, \qquad \omega_{-i} = -\frac{\partial_i H}{2\Omega H^{\alpha+1}}\,e^+\,,\\
\omega_{+i} &= \frac{\partial_i\mathcal{G}}{2\Omega H^{\alpha-1}}\,e^+ - \frac{\partial_i H}{2\Omega H^{\alpha+1}}\,e^- + \left(H\partial_{(i}a_{j)} - \frac{1}{\Omega H^{\alpha-1}}\tilde\nabla\left(\Omega H^\alpha\right)\delta_{ij}\right)\,e^j\,,\\
\omega_{ij} &= -H\partial_{[i}a_{j]}e^+ + \frac{2}{\Omega^2 H^{2\alpha}}\delta_{k[i}\partial_{j]}\left(\Omega H^\alpha\right)\,e^k\,.
\end{aligned}
\tag{D.1}
$$

Riemann tensor:

$$
\begin{aligned}
R_{+-} &= \frac{\partial_i H\partial_i H}{4\Omega^2 H^{2\alpha+2}}\,e^+\wedge e^- - \frac{1}{2\Omega H^{\alpha-1}}\times\\
&\quad\times\left[\tilde\nabla\partial_i\left(\ln H\right) - \partial_i\left(\ln H\right)\tilde\nabla\left(\ln\Omega H^\alpha\right) + \partial_{[k}a_{i]}\partial_k\left(\ln H\right)\right]\,e^+\wedge e^i\,,
\end{aligned}
\tag{D.2}
$$

$$
\begin{aligned}
R_{-i} &= \frac{1}{2\Omega^2 H^{2\alpha}}\left[\frac{1}{H}\partial_i\partial_j H - 2\partial_{(i}\left(\ln H\right)\partial_{j)}\left(\ln\Omega\right) - \left(2\alpha+\frac{3}{2}\right)\partial_i\left(\ln H\right)\partial_j\left(\ln H\right) + \right.\\
&\quad\left. + \delta_{ij}\partial_k\left(\ln H\right)\partial_k\left(\ln\Omega H^\alpha\right)\right]\,e^+\wedge e^j\,.
\end{aligned}
\tag{D.3}
$$

(++) **component of the Ricci tensor:**

$$
\begin{aligned}
R_{++} &= \frac{1}{2\Omega^2 H^{2\alpha-1}}\left(-\Delta\mathcal{G} + \partial_k\mathcal{G}\partial_k\ln H\right) +\\
&\quad + H^2\left(-2\tilde\nabla\tilde\nabla\ln\Omega H^\alpha - 2\tilde\nabla\ln\Omega H^{\alpha+1}\tilde\nabla\ln\Omega H^\alpha + \left(\partial_k a^k\right)\tilde\nabla\ln\Omega^2 H^{2\alpha+1} + \right.\\
&\quad\left. + \tilde\nabla\partial_k a^k - \frac{1}{2}(\partial_i a^j)(\partial^i a_j) - \frac{1}{2}(\partial_i a^j)(\partial_j a^i)\right)\,.
\end{aligned}
\tag{D.4}
$$

E. Integrability conditions and Einstein equations

The Killing spinor equation (2.9) implies the integrability condition

$$[\mathcal{D}_\nu, \mathcal{D}_\mu]\epsilon = \Bigg[\frac{1}{\ell}({}^*\mathcal{F}_{\nu\mu}\Gamma_5 - i\mathcal{F}_{\nu\mu}) + \frac{1}{2\ell^2}\Gamma_{\nu\mu} + \frac{1}{4}\mathcal{R}^{ab}{}_{\nu\mu}\Gamma_{ab} - \mathcal{F}^{\alpha\beta}\mathcal{F}_{\beta[\nu}\Gamma_{\mu]\alpha} + \\ + \frac{1}{4}\mathcal{F}_{\alpha\beta}\mathcal{F}^{\alpha\beta}\Gamma_{\nu\mu} - \frac{i}{\ell}\mathcal{F}^{\alpha}{}_{[\nu}\Gamma_{\mu]\alpha} - \frac{i}{2}\Gamma_{\alpha\beta[\nu}\nabla_{\mu]}\mathcal{F}^{\alpha\beta} - i\nabla_{[\nu}\mathcal{F}_{\mu]}{}^{\alpha}\Gamma_\alpha\Bigg]\epsilon = 0\,. \tag{E.1}$$

Contracting this with Γ^μ and using the Bianchi identity $R_{\nu[\mu\alpha\beta]} = 0$, we obtain

$$E_{\nu\alpha}\Gamma^\alpha\epsilon + i\nabla_\mu\mathcal{F}^{\mu\alpha}\Gamma_\alpha\Gamma_\nu\epsilon + \frac{1}{2}\left[\nabla^{[\sigma}\mathcal{F}^{\alpha\beta]}\epsilon_{\sigma\alpha\beta\nu}\Gamma_5 + 3i\nabla_{[\nu}\mathcal{F}_{\alpha\beta]}\Gamma^{\alpha\beta}\right]\epsilon = 0\,, \tag{E.2}$$

where we defined

$$E_{\nu\alpha} = R_{\nu\alpha} + \frac{3}{\ell^2}g_{\nu\alpha} + 2\mathcal{F}_{\nu\beta}\mathcal{F}^{\beta}{}_{\alpha} + \frac{1}{2}\mathcal{F}^2 g_{\nu\alpha}\,. \tag{E.3}$$

Let us assume that the Maxwell equations and the Bianchi identity for $\mathcal{F}$ hold. We then conclude that

$$E_{\nu\alpha}\Gamma^\alpha\epsilon = 0\,. \tag{E.4}$$

If we multiply this from the left with $\bar\epsilon$ we get

$$E_{\nu\alpha}V^\alpha = 0\,. \tag{E.5}$$

On the other hand, multiplying with $E_{\nu\sigma}\Gamma^\sigma$ yields

$$E_{\nu\sigma}E_\nu{}^\sigma = 0\,, \tag{E.6}$$

where we do not sum over the index ν. One can now proceed analogously to [4] and use (E.5) and (E.6) to show that in the timelike case the Killing spinor equations (plus the Maxwell equations and Bianchi identity for $\mathcal{F}$) imply the Einstein equations $E_{\nu\alpha} = 0$, whereas in the lightlike case the component $E_{++} = 0$ must be additionally imposed.

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